

THE L^p -DIAMETER OF THE SPACE OF CONTRACTIBLE LOOPS

MICHAEL BRANDENBURSKY AND EGOR SHELUKHIN

ABSTRACT. We prove that the space of contractible simple loops of a given fixed area in any compact oriented surface has infinite diameter as a homogeneous space of the group of area-preserving diffeomorphisms endowed with the L^p -metric. As a special case, this resolves the L^p -metric analogue of the well-known question in symplectic topology regarding the space of equators on the two-sphere. Our methods involve a new class of functionals on a normed group, which are more general than quasi-morphisms.

1. INTRODUCTION AND MAIN RESULTS

Consider the two-sphere S^2 with the standard area form and the space \mathcal{E} of equators in it: smooth embedded loops dividing the sphere into two components of equal areas. The identity component \mathcal{G} of the group area-preserving diffeomorphisms acts transitively on \mathcal{E} . This allows us to induce a pseudo-metric on \mathcal{E} given a metric on \mathcal{G} . A remarkable bi-invariant such metric on \mathcal{G} was introduced by Hofer [14] and subsequently Viterbo, Polterovich, and Lalonde-McDuff [17]. The induced pseudo-metric on \mathcal{E} is non-degenerate, and is therefore a metric, as was proven by Chekanov [8]. The first interesting invariant of a metric space is its diameter. Polterovich [19] proved that the diameter of \mathcal{G} in Hofer's metric is infinite, a result which was recently extended to provide quasi-isometric embeddings of linear spaces of arbitrarily large dimension [9, 20]. However, the diameter of \mathcal{E} in the Chekanov-Hofer metric is still unknown at the time of writing, see [18, Problem 32].

In this paper we show that the diameter of \mathcal{E} is infinite in the metric induced from the right-invariant L^p -metric on \mathcal{G} . The case $p = 2$ is closely related to questions in hydrodynamics, while the case $p = 1$ has the following intuitive definition (see [7]). The L^1 -length $\ell_1(\{\phi_t\})$ of an isotopy $\{\phi_t\}$ is the average length of a trajectory $\{\phi_t(x)\}$ of a point x , and the L^1 -norm of ϕ is defined as

$$|\phi|_1 = \inf_{\phi_1=\phi} \ell_1(\{\phi_t\}).$$

It is easy to see that this norm is non-degenerate and defines the right invariant metric $d_1(\phi, \psi) = |\psi\phi^{-1}|_1$. This metric induces a pseudo-metric on \mathcal{E} , given by

$$d_1(L, L') = \inf_{\phi(L)=L'} |\phi|_1.$$

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This pseudo-metric is non-degenerate as shown in Proposition 2 below, and therefore defines a metric. Our main result concerning the resulting metric space is as follows.

Theorem A. *The diameter of \mathcal{E} equipped with the metric d_1 is infinite.*

It is easy to see (see for example [7, Remark 1.1]) that in this case the diameter of \mathcal{E} in d_p for all $p \geq 1$ is also infinite. Moreover, in Theorem B we generalize this result to the space \mathcal{E}_a of all embedded loops bounding a disk of area $a \in (0, \text{Area}(\Sigma))$ in every closed oriented surface Σ with an area form (see also Remark 6). We call these loops equators as well. In fact the argument proving Theorem A extends verbatim to \mathcal{E}_a , with the caveat that for the sphere $\mathcal{E}_a = \mathcal{E}_{A-a}$ where $A = \text{Area}(S^2)$.

The methods of the proof involve functionals obtained from quasi-morphisms on braid groups in the spirit of [13], see for instance [6, 7]. However, our functionals are *not* quasi-morphisms on \mathcal{G} as it has usually been in previous work in the field. Instead, they satisfy weaker properties, which, however, are sufficient for our purposes. Thus, we introduce the notion of a *para-morphism* on a metric group. We now outline our approach in slightly greater detail.

1.1. Outline of the proof. First, we show, using integration along the configuration space of points in S^2 , and crucially [6] for the first point, the following result.

Proposition 1. *For each equator L there is a functional*

$$\Phi : \mathcal{G} \rightarrow \mathbb{R}$$

satisfying the following properties for A, B, C, D depending L and the geometry of S^2 :

- (1) $|\Phi(gf) - \Phi(f) - \Phi(g)| \leq C + D|g|_1$ for all $f, g \in \mathcal{G}$
- (2) $\bar{\Phi}(f) = \liminf_{k \rightarrow \infty} \Phi(f^k)/k$ is well-defined on \mathcal{G} and not identically zero
- (3) $|\Phi(h)| \leq B$ for all $h \in \mathcal{G}$ such that $h(L) = L$, and in particular $\bar{\Phi}(h) = 0$
- (4) $|\Phi(f)| \leq A(|f|_1 + 1)$ for all $f \in \mathcal{G}$

Note that if $D = 0$ then Φ is a quasi-morphism on \mathcal{G} . We therefore call a functional satisfying the properties 1, 3, 4 an L^1 *para-morphism*. In addition, Property 1 implies that $\delta\Phi$ is a $(1, 1)$ -bounded 2-group cocycle and defines a certain cohomology class, see e.g., [11, 12]. The notion of a para-morphism can be generalized to the general framework on a normed group (G, ν) , where ν is the norm, and we expect it to yield further applications in geometry and dynamics. We remark that we construct such Φ using surface braid groups on n strands for each $n > 3$ and denote it Φ_n .

We now explain how Proposition 1 implies Theorem A.

Proof of Theorem A. Let L be the standard equator and Φ provided by Proposition 1. Now suppose that $g(L) = f(L)$ for $f, g \in \mathcal{G}$. Then $h = g^{-1}f$ satisfies $h(L) = L$. Therefore,

$$|\Phi(h)| \leq B$$

by Property 3. From Properties 1 and 4 we obtain

$$|\Phi(f)| \leq |\Phi(h)| + |\Phi(g^{-1})| + C + D|g^{-1}|_1 \leq A + B + C + (A + D)|g|_1.$$

Taking infimum over all $g \in \mathcal{G}$ with $g(L) = f(L)$ yields

$$(1) \quad |\Phi(f)| \leq E + F d_1(L, f(L))$$

for constants $E = A + B + C$, $F = A + B$. Now Property 2 implies that there is a sequence $f_i \in \mathcal{G}$ such that $|\Phi(f_i)| \rightarrow \infty$ as $i \rightarrow \infty$ and hence by (1)

$$d_1(L, f_i(L)) \rightarrow \infty$$

as well. This finishes the proof. \square

1.2. Organization of the paper. In Section 3 we prove Proposition 1 and in Section 4 we explain how the arguments generalize to prove analogues of this proposition for \mathcal{E}_a , where $0 < a < \text{Area}(\Sigma)$ and Σ is a closed oriented surface, and hence extend Theorem A to this setting.

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2. PRELIMINARIES

Recall that an alternative way to define the L^1 -length is

$$\ell_1(\{\phi_t\}) = \int_0^1 \int_{\Sigma} |X_t| \omega dt$$

where X_t is the time-dependent vector field generating ϕ_t and ω is the area form on Σ . Note that for a constant C_0 depending only on the geometry of Σ we have

$$\ell_1(\{\phi_t\}) \leq C_0 \int_0^1 \int_{\Sigma} |d(H_t)| \omega dt,$$

where $H_t(x) = H(t, x)$ is the time-dependent Hamiltonian generating X_t .

Proposition 2. *Let L be an embedded separating loop on a compact orientable surface Σ . Let \mathcal{E}_L be the orbit of L under the identity component \mathcal{G} of the group of area-preserving diffeomorphisms. Then the metric d_1 on \mathcal{E}_L induced by the L^1 metric on \mathcal{G} is non-degenerate.*

By [7, Remark 1.1], Proposition 2 applies to all L^p metrics for $p \geq 1$.

Proof. Let A and B be the two connected components of the complement of L . Then $\phi L \neq L$ if and only if $\phi(A) \cap B \neq \emptyset$. Let U be a connected component of the non-empty open set $\phi(A) \cap B$ and $D = D(z, \varepsilon)$ be a metric ball of radius $\varepsilon > 0$ in U , such that $D(z, 2\varepsilon)$ is also in U . Note that $U \subset B$ while $\phi^{-1}U \subset A$. Therefore, $\phi^{-1}(U) \cap U = \emptyset$ and moreover $d(\phi^{-1}x, x) > \varepsilon$ for all $x \in D$. Thus

$$|\phi|_1 = |\phi^{-1}|_1 \geq \varepsilon \cdot \text{Area}(D)$$

and hence taking infimum over ϕ with $\phi(L) = L'$ we get

$$d_1(L', L) \geq \varepsilon \cdot \text{Area}(D) > 0.$$

□

3. PROOF OF PROPOSITION 1

Without loss of generality let L be the standard equator. We proceed in a number of steps starting with a definition of the functional Φ_n .

3.1. Definition of the para-morphism. Let $n \in \mathbb{N}$ such that $n > 3$, and let $C_n(S^2)$ be the configuration space of all unordered n -tuples of pairwise distinct points in S^2 . Recall that the Birman map (see e.g. [10]):

$$\text{Push}: B_n(S^2) \rightarrow \text{MCG}(S^2, n),$$

where $B_n(S^2) = \pi_1(C_n(S^2), z)$ is the spherical braid group on n strands and $\text{MCG}(S^2, n)$ is the mapping class group of the n -punctured sphere, is defined as follows: let $\alpha(t)$, $t \in [0, 1]$, be a loop in $C_n(S^2)$ based at z and $h_t \in \text{Diff}(S^2)$ an isotopy such that $h_t(z) = \alpha(t)$. We define $\text{Push}(\alpha) := [h_1]$ where α is the braid represented by the loop $\alpha(t)$. The braid α is called *reducible* if $\text{Push}(\alpha)$ is a reducible mapping class. We denote by $Q_{\text{BF}}(B_n(S^2))$ the space of homogeneous quasimorphisms on $B_n(S^2)$ which vanish on reducible braids. It follows from the celebrated paper by Bestvina and Fujiwara [1] that the space $Q_{\text{BF}}(B_n(S^2))$ is infinite dimensional, see [5, Section 4.A.].

Let $\{f_t\}$ be an isotopy in \mathcal{G} from the identity to $f \in \mathcal{G}$. For $x, y \in S^2$ we define a loop $\gamma_{x,y}: [0, 1] \rightarrow S^2$ by

$$\gamma_{x,y}(t) := \begin{cases} \alpha_{3t} & \text{for } t \in [0, \frac{1}{3}] \\ f_{3t-1}(x) & \text{for } t \in [\frac{1}{3}, \frac{2}{3}] \\ \beta_{3t-2} & \text{for } t \in [\frac{2}{3}, 1], \end{cases}$$

where α_t is a shortest path on S^2 from y to x , and β_t is a shortest path on S^2 from $f(x)$ to y .

Let $X_n(S^2)$ be the configuration space of all ordered n -tuples of pairwise distinct points in the two-sphere S^2 . Its fundamental group $\pi_1(X_n(S^2), z)$ is identified with the spherical pure braid group $P_n(S^2)$, where $z = (z_1, \dots, z_n)$ in $X_n(S^2)$ is a base point. For almost every $x = (x_1, \dots, x_n) \in X_n(S^2)$ the n -tuple of loops $(\gamma_{x_1, z_1}, \dots, \gamma_{x_n, z_n})$ is a based loop in the configuration space $X_n(S^2)$. Let $\gamma(\{f_t\}, x) \in P_n(S^2) = \pi_1(X_n(S^2), z)$ be the element represented by this loop, and let $\varphi: P_n(S^2) \rightarrow \mathbf{R}$ be a homogeneous quasimorphism. Since the group $\pi_1(\text{Ham}(S^2))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the number $\varphi(\gamma(\{f_t\}, x))$ does not depend on the choice of the isotopy $\{f_t\}$ and from now on is denoted by $\varphi(\gamma(f, x))$. Note that for $f, g \in \mathcal{G}$ we have the following cocycle condition:

$$(2) \quad \varphi(\gamma(gf, x)) = \varphi(\gamma(g, f(x))) \cdot \varphi(\gamma(f, x)).$$

Denote by D_+ and D_- the open Northern and Southern hemisphere respectively. Consider the subspace $X_n(D_+ \cup D_-) \subset X_n(S^2)$ consisting of those n -tuples points in S^2 where either all points lie in D_+ , or all points lie in D_- , or exactly $n-1$ points lie in D_+ and one point in D_- , or exactly $n-1$ points lie in D_- and one point in D_+ . Let $\varphi \in Q_{\text{BF}}(B_n(S^2))$ a nontrivial homogeneous quasimorphism. We define $\Phi_n: \mathcal{G} \rightarrow \mathbb{R}$ as follows:

$$(3) \quad \Phi_n(f) := \int_{X_n(S^2)} \varphi(\gamma(f, x)) dx - \int_{X_n(D_+ \cup D_-)} \varphi(\gamma(f, x)) dx.$$

3.2. Property 1. Let $0 < k < n$ and $X_{n,k}(D_+) \subset X_n(S^2)$ the subspace where in each n -tuple of points exactly k points lie in D_+ . As usual, we denote by D_φ the defect of the quasimorphism φ . Now,

$$\begin{aligned} |\Phi_n(gf) - \Phi_n(f) - \Phi_n(g)| &\leq \sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(gf, x)) - \varphi(\gamma(g, x)) - \varphi(\gamma(f, x))| dx \leq \\ &\sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(gf, x)) - \varphi(\gamma(g, f(x))) - \varphi(\gamma(f, x))| dx + \\ &\sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(g, f(x))) - \varphi(\gamma(g, x))| dx \leq \\ &\text{vol}(X_n(S^2))D_\varphi + 2 \int_{X_n(S^2)} |\varphi(\gamma(g, x))| dx \leq \text{vol}(X_n(S^2))D_\varphi + 2(C' + D'|g|_1), \end{aligned}$$

where the last inequality follows immediately from [7, Theorem 1] (see also [6, Theorem 1]). By setting

$$C := (\text{vol}(X_n(S^2))D_\varphi + 2C'), D := 2D'$$

we obtain Property 1.

3.3. Property 4. Let $f \in \mathcal{G}$. Then as before

$$|\Phi_n(f)| \leq \sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(f, x))| dx \leq \int_{X_n(S^2)} |\varphi(\gamma(f, x))| dx \leq C' + D'|f|_1.$$

By setting $A := \max\{C', D'\}$ we obtain Property 4.

3.4. A fragmentation result.

Lemma 3. *Let L be an embedded contractible loop in a surface Σ and A, B connected open sets such that $\Sigma \setminus L = A \sqcup B$. Suppose that ϕ is the time-one map of a compactly supported Hamiltonian isotopy such that $\phi(L) = L$. Then there is a constant K depending on L only, such that $d_1(\phi, fg) < K$ for some $f \in \text{Ham}_c(A)$ and $g \in \text{Ham}_c(B)$.*

Proof. First $f = \phi|_L : L \rightarrow L$ is a diffeomorphism connected to the identity or not. If $\Sigma \neq S^2$ then f is connected to the identity, as it has degree 1. Indeed, in this case, the two connected components of $\Sigma \setminus L$ are not diffeomorphic and therefore are both preserved under ϕ . In particular we may consider ϕ as a self-map of the pair (D, L) where D is the closure of the disk connected component of $\Sigma \setminus L$. Now $\deg(f) = 1$ as $\deg(\phi) = 1$ and the isomorphism $H_2(D, L; \mathbb{Z}) \rightarrow H_1(L; \mathbb{Z})$ coming from the long exact sequence of a pair is functorial.

If $\Sigma = S^2$ and f is not connected to the identity, we compose ϕ with a rotation ρ of S^2 by angle π along an axis passing through L . Then $\phi' = \phi\rho$ satisfies $\phi'(L) = L$, $f' = \phi'|_L : L \rightarrow L$ isotopic to the identity and $|\phi'|_1 \leq |\phi|_1 + c$ for c depending on the geometry of S^2 only.

Therefore, we may assume that f is isotopic to the identity. Let $\{f_t\}$ for $f_t \in \text{Diff}(L)$ be an isotopy with time-one map f . Lift $\{f_t\}$ to a Hamiltonian isotopy $\{\psi_t\}$ of T^*L with Hamiltonian $F(t, x)$ 1-homogeneous in the momentum variable and vanishing on the zero section L for all t . Identify L with S^1 and T^*L with $\mathbb{R} \times S^1$. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a cutoff function such that $\chi(x) = 1$ for $|x| \leq 1/3$, $\text{supp}(\chi) \subset (-1, 1)$ and $|\chi'| \leq 2$ everywhere. For $\delta > 0$ set $\chi_\delta(x) = \chi(x/\delta)$. Then $\chi_\delta(x) = 1$ for $|x| \leq \delta/3$, $\text{supp}(\chi_\delta) \subset (-\delta, \delta)$ and $|\chi'_\delta| \leq 2\delta^{-1}$ everywhere. Finally, consider χ_δ as a function on $\mathbb{R} \times S^1 \cong T^*L$, set $G(t, x) = F(t, x)\chi_\delta(x)$, and let $\{\psi_t^\delta\}$ be the flow of G . Then, for all t , $\psi_t^\delta|_L = f_t$ and $\text{supp}(\psi_t^\delta) \subset (-\delta, \delta) \times S^1$. The key property of this construction is the following.

Claim 4. *As $\delta \rightarrow 0$, $\ell_1(\{\psi_t^\delta\}) \rightarrow 0$.*

Let us prove this statement. As $F(t, x) = 0$ for all $x = (0, q)$ and all t , there exists a function $H(t, x)$ such that $F(t, p, q) = pH(t, p, q)$. Hence $\chi_\delta(p)F(t, p, q) = \chi_\delta(p)pH(t, p, q)$. Therefore,

$$d(\chi_\delta(p)F(t, p, q)) = \chi'_\delta(p)pH(t, p, q)dp + \chi_\delta(p)H(t, p, q)dp + \chi_\delta(p)p dH_t(p, q).$$

Now as $\delta \rightarrow 0$, taking the size of the support of χ_δ into account

$$\chi'_\delta(p)pH(t, p, q)dp = O(\delta^{-1}\delta) = O(1),$$

$$\chi_\delta(p)H(t, p, q)dp = O(1),$$

$$\chi_\delta(p)pdH_t(p, q) = O(\delta)$$

and therefore

$$\ell_1(\{\psi_t^\delta\}) = 2\delta(O(1) + O(1) + O(\delta)) = O(\delta).$$

This finishes the proof.

Finally, identify a neighborhood U of L in Σ with a neighborhood V of L in T^*L . Fix $\varepsilon > 0$ and consider δ sufficiently small so that $\text{supp}(\psi_t^\delta)$ is contained in V for all t and $\{\psi_t^\delta\}$ corresponds to an isotopy $\{\phi_t^\delta\}$ on Σ supported in U such that $|\phi_1^\delta| \leq \ell(\{\phi_t^\delta\}) < \varepsilon$. Then $(\phi_1^\delta)^{-1}\phi = fg$ for some $f, g \in \mathcal{G}$ such that $\text{supp}(f) \subset \bar{A}$ and $\text{supp}(g) \subset \bar{B}$ and $d_1(\phi, (\phi_1^\delta)^{-1}\phi) = |\phi_1^\delta| < \varepsilon$. Finally, rescaling f, g a little by the Liouville flow in a neighborhood of \bar{A} and \bar{B} towards the skeleta of A, B we obtain f', g' such that $\text{supp}(f') \subset A$ and $\text{supp}(g') \subset B$ and $d_1(fg, f'g') < \varepsilon$. In total, we obtain

$$d_1(\phi, f'g') < 2\varepsilon$$

for $\phi|_L$ isotopic to the identity. Setting $K := c + 2\varepsilon$, where we set $c = 0$ whenever $\Sigma \neq S^2$, we conclude the proof of the lemma for a general ϕ . \square

3.5. Property 3. Let $h \in \mathcal{G}$ such that $h(L) = L$. First, we show that there exists a constant $C_1 \in \mathbb{R}$ such that for every $f \in \text{Ham}_c(D_+)$ and $g \in \text{Ham}_c(D_-)$ we have $|\Phi_n(fg)| \leq C_1$.

Let $\{f_t\}$ be an isotopy in $\text{Ham}_c(D_+)$ between $f_0 = \mathbb{1}$ and $f_1 = f$, and $\{g_t\}$ be an isotopy in $\text{Ham}_c(D_-)$ between $g_0 = \mathbb{1}$ and $g_1 = g$. Then we have the following braid identity

$$\gamma(\{f_t * g_t\}, x) = \gamma(\{f_t\}, g(x)) \cdot \gamma(\{g_t\}, x).$$

Since $\{f_t\} \in \text{Ham}_c(D_+)$ and $\{g_t\} \in \text{Ham}_c(D_-)$, then for every $1 < k < n - 1$ and each $x \in X_{n,k}(D_+)$ the braids $\gamma(\{f_t\}, g(x))$ and $\gamma(\{g_t\}, x)$ are reducible. Hence $\varphi(\gamma(f, g(x))) = 0$ and $\varphi(\gamma(g, x)) = 0$. It follows that

$$|\varphi(\gamma(fg, x))| \leq D_\varphi.$$

Setting $C_1 := D_\varphi \text{vol}(X_n(S^2))$, we obtain

$$(4) \quad |\Phi_n(fg)| \leq \sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(fg, x))| dx \leq C_1.$$

We proceed with the proof of Property 3. It follows from Lemma 3 that there is a constant K depending on L only, such that $d_1(h, fg) < K$ for some $f \in \text{Ham}_c(D_+)$ and $g \in \text{Ham}_c(D_-)$. Now,

$$\begin{aligned} |\Phi_n(h) - \Phi_n(fg)| &\leq |\Phi_n(h) - \Phi_n(fg) - \Phi_n(h(fg)^{-1})| + |\Phi_n(h(fg)^{-1})| \leq \\ &C + D|h(fg)^{-1}|_1 + |\Phi_n(h(fg)^{-1})| \leq C + D|h(fg)^{-1}|_1 + (A + 1)|h(fg)^{-1}|_1 = \\ &C + (D + A + 1)|h(fg)^{-1}|_1 \leq C + (D + A + 1)K, \end{aligned}$$

where the second inequality follows from Property 1, and the third inequality follows from Property 4. Hence by (4)

$$|\Phi_n(h)| \leq |\Phi_n(fg)| + C + (D + A + 1)K \leq C_1 + C + (D + A + 1)K.$$

We conclude the proof by denoting $B := C_1 + C + (D + A + 1)K$.

3.6. Property 2. First, we show that for each $f \in \mathcal{G}$ the sequence $\{\Phi_n(f^k)/k\}_{k=1}^\infty$ is bounded, which in turn implies that $\bar{\Phi}_n(f)$ is well-defined. It follows from [5, Lemma 2.1] that there exists a constant K_1 such that for almost every $x \in X_n(S^2)$ we have $|\varphi(\gamma(f, x))| \leq K_1$. Cocycle condition 2 implies that

$$\varphi(\gamma(f^k, x)) = \varphi(\gamma(f, f^{k-1}(x))) \cdot \dots \cdot \varphi(\gamma(f, x)).$$

Hence

$$|\varphi(\gamma(f^k, x))|/k \leq K_1 + D_\varphi.$$

The above inequality yields

$$\begin{aligned} |\Phi_n(f^k)/k| &\leq \sum_{k=2}^{n-2} \int_{X_{n,k}(D_+)} |\varphi(\gamma(f^k, x))|/k \, dx \leq \\ &\int_{X_n(S^2)} |\varphi(\gamma(f^k, x))|/k \, dx \leq (K_1 + D_\varphi) \text{vol}(X_n(S^2)), \end{aligned}$$

which implies that $\bar{\Phi}_n(f)$ is well-defined.

Now we prove that there exists $h \in \mathcal{G}$ such that $\bar{\Phi}_n(h) \neq 0$. For the simplicity we prove this fact for $n = 4$. Let $0 \neq \varphi \in Q_{\text{BF}}(B_4(S^2))$ and $\beta \in P_4(S^2) < B_4(S^2)$ such that $\varphi(\beta) = b \neq 0$. By the Ishida construction [15] (see also [3, 5]), there exist four embedded discs D_1, D_2, D_3, D_4 in S^2 and $f \in \mathcal{G}$ such that:

- $D_1, D_2 \subset D_+, D_1 \cap D_2 = \emptyset, D_3, D_4 \subset D_-, D_3 \cap D_4 = \emptyset, a_i := \text{area}(D_i)$.
- For each $k \in \mathbb{Z}$, each permutation $\sigma \in S_4$ and each $x = (x_1, x_2, x_3, x_4) \in X_4(S^2)$ such that $x_i \in D_{\sigma(i)}$ we have $\varphi(\gamma(f^k, x)) = kb$.

By definition $\bar{\Phi}_4(f) = \lim_{p \rightarrow \infty} \Phi_4(f^{k_p})/k_p$ for some increasing sequence $\{k_p\}_{p=1}^\infty$. Thus

$$\begin{aligned} \bar{\Phi}_4(f) &= \lim_{p \rightarrow \infty} \Phi_4(f^{k_p})/k_p = \lim_{p \rightarrow \infty} \int_{X_{4,2}(D_+)} \frac{\varphi(\gamma(f^{k_p}, x))}{k_p} \, dx = \\ &\lim_{p \rightarrow \infty} \left(\int_{X_4(D_1 \cup \dots \cup D_4)} \frac{\varphi(\gamma(f^{k_p}, x))}{k_p} \, dx + \int_{X_{4,2}(D_+) \setminus X_4(D_1 \cup \dots \cup D_4)} \frac{\varphi(\gamma(f^{k_p}, x))}{k_p} \, dx \right), \end{aligned}$$

where $X_4(D_1 \cup \dots \cup D_4)$ consists of those 4-tuples $x = (x_1, x_2, x_3, x_4) \in X_4(S^2)$ where each x_i lies in one of D_j . Note that construction of f yields

$$(5) \quad \lim_{p \rightarrow \infty} \int_{X_4(D_1 \cup \dots \cup D_4)} \frac{\varphi(\gamma(f^{kp}, x))}{k_p} dx = 4!(ba_1a_2a_3a_4 + c_{1133}a_1^2a_3^2 + c_{1144}a_1^2a_4^2 + c_{1134}a_1^2a_3a_4 + c_{1233}a_1a_2a_3^2 + c_{1244}a_1a_2a_4^2 + c_{2233}a_2^2a_3^2 + c_{2234}a_2^2a_3a_4 + c_{2244}a_2^2a_4^2),$$

where $c_{ijkl} := \varphi(\gamma(f, (x_i, x_j, x_k, x_l)))$ such that $x_i \in D_i, x_j \in D_j, x_k \in D_k, x_l \in D_l$.

The expression in 5 is a homogeneous polynomial $P(a_1, a_2, a_3, a_4)$ in variables a_1, a_2, a_3, a_4 , i.e., $P(ra_1, ra_2, ra_3, ra_4) = r^4 P(a_1, a_2, a_3, a_4)$. Without loss of generality we assume that $\text{area}(S^2) = 1$. Since polynomial P is non-trivial, there exist $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and $0 \neq c \in \mathbb{R}$ such that $a := a_1 + a_2 + a_3 + a_4 < 1$ and $P(a_1, a_2, a_3, a_4) = c$. Again by Ishida construction, for each $0 < r < \frac{1}{a}$ there exist four embedded discs $D_{1,r}, D_{2,r}, D_{3,r}, D_{4,r}$ in S^2 and $f_r \in \mathcal{G}$ such that:

- $D_{1,r}, D_{2,r} \subset D_+$, $D_{1,r} \cap D_{2,r} = \emptyset$, $D_{3,r}, D_{4,r} \subset D_-$, $D_{3,r} \cap D_{4,r} = \emptyset$, $ra_i = \text{area}(D_{i,r})$.
- For each $k \in \mathbb{Z}$, each permutation $\sigma \in S_4$ and each $x = (x_1, x_2, x_3, x_4) \in X_4(S^2)$ such that $x_i \in D_{\sigma(i),r}$ we have $\varphi(\gamma(f_r^k, x)) = kb$.

We obtain

$$\bar{\Phi}_4(f_r) = \lim_{p \rightarrow \infty} \Phi_4(f_r^{k_{r,p}})/k_{r,p} = r^4 c + \lim_{p \rightarrow \infty} \int_{X_{4,2}(D_+) \setminus X_4(D_{1,r} \cup \dots \cup D_{4,r})} \frac{\varphi(\gamma(f_r^{k_{r,p}}, x))}{k_{r,p}} dx.$$

There exists $d \in \mathbb{N}$ and $A_i \in P_n(S^2)$ the standard Artin generator for each $1 \leq i \leq d$ such that $\beta = A_1 \dots A_d$. Again, by Ishida construction

$$f_r = f_{1,r} \circ \dots \circ f_{d,r},$$

where each $f_{i,r} \in \mathcal{G}$ is a Morse autonomous diffeomorphism (on its support). Moreover, for almost every $x \in X_4(S^2)$ we have

$$\gamma(f_{i,r}, x) = \alpha'_{f_{i,r},x} \delta_{f_{i,r},x} \alpha''_{f_{i,r},x}$$

such that $\delta_{f_{i,r},x}$ is a commuting product of reducible braids and the length of braids $\alpha'_{f_{i,r},x}, \alpha''_{f_{i,r},x}$ is universally bounded by a constant which does not depend on r and x , see e.g. [2, proof of Theorem 4.5]. It follows that there exists $K_1 > 0$ such that for almost every $x \in X_4(S^2)$ and every $0 < r < \frac{1}{a}$ we have

$$|\varphi(\gamma(f_{i,r}, x))| \leq K_1.$$

It follows that

$$\left| \frac{\varphi(\gamma(f_r^{k_{r,p}}, x))}{k_{r,p}} \right| \leq d(K_1 + D_\varphi).$$

Thus

$$\left| \lim_{p \rightarrow \infty} \int_{X_{4,2}(D_+) \setminus X_4(D_{1,r} \cup \dots \cup D_{4,r})} \frac{\varphi(\gamma(f_r^{k_{r,p}}, x))}{k_{r,p}} dx \right| \leq d(K_1 + D_\varphi) \text{vol}(X_{4,2}(D_+) \setminus X_4(D_{1,r} \cup \dots \cup D_{4,r})).$$

Since $\lim_{r \rightarrow \frac{1}{a}} \text{vol}(X_{4,2}(D_+) \setminus X_4(D_{1,r} \cup \dots \cup D_{4,r})) = 0$, we obtain

$$\lim_{r \rightarrow \frac{1}{a}} \bar{\Phi}_4(f_r) = \frac{c}{a^4} \neq 0.$$

It follows that there exists r such that $\bar{\Phi}_4(f_r) \neq 0$. By setting $h := f_r$ we finish the proof of Property 2.

4. GENERAL LOOPS AND SURFACES

Let Σ be a closed oriented surface of positive genus with an area form ω . Let \mathcal{G}_Σ be the identity component of the group of area-preserving diffeomorphisms of Σ .

Theorem B. *Let $a \in (0, \text{Area}(\Sigma))$. Then the diameter of \mathcal{E}_a equipped with d_1 is infinite.*

Remark 5. Let $\text{Ham}(\Sigma) < \mathcal{G}_\Sigma$ the group of Hamiltonian diffeomorphisms of Σ . Theorem B implies that the diameter of \mathcal{E}_a equipped with the d_1 -metric coming from $\text{Ham}(\Sigma)$ is infinite.

Let $n > 3$ and L a equator in Σ . Note that in order to prove Theorem B it is enough to construct a functional

$$\Phi_{\Sigma,n} : \mathcal{G}_\Sigma \rightarrow \mathbb{R}$$

which satisfies properties 1, 2, 3 and 4 in the case of \mathcal{G}_Σ .

The construction of $\Phi_{\Sigma,n}$ is very similar to the construction of Φ_n presented in Subsection 3.1. Let $n \in \mathbb{N}$ such that $n > 3$, and let $C_n(\Sigma)$ be the configuration space of all unordered n -tuples of pairwise distinct points in Σ . Recall the Birman Push map (see e.g. [10]):

$$\text{Push} : B_n(\Sigma) \rightarrow \text{MCG}(\Sigma, n),$$

where $B_n(\Sigma) = \pi_1(C_n(\Sigma), z)$ is the surface braid group on n strands and $\text{MCG}(\Sigma, n)$ is the mapping class group of the n -punctured surface Σ . This map is injective when Σ is a hyperbolic surface. In case when Σ is a torus T^2 this map has a kernel which equals to the center $Z(B_n(T^2)) \cong \mathbb{Z}^2$. The braid α is called *reducible* if $\text{Push}(\alpha)$ is a reducible mapping class. We denote by $Q_{\text{BF}}(B_n(\Sigma))$ the space of homogeneous quasimorphisms on $B_n(\Sigma)$ which vanish on reducible braids. It follows from the celebrated paper by Bestvina and Fujiwara [1] that the space $Q_{\text{BF}}(B_n(\Sigma))$ is infinite dimensional, see [5, Section 4.A.].

Let $\{f_t\}$ be an isotopy in \mathcal{G}_Σ from the identity to $f \in \mathcal{G}_\Sigma$. For $x, y \in \Sigma$ we define a loop $\gamma_{x,y}: [0, 1] \rightarrow \Sigma$ by

$$\gamma_{x,y}(t) := \begin{cases} \alpha_{3t} & \text{for } t \in [0, \frac{1}{3}] \\ f_{3t-1}(x) & \text{for } t \in [\frac{1}{3}, \frac{2}{3}] \\ \beta_{3t-2} & \text{for } t \in [\frac{2}{3}, 1], \end{cases}$$

where α_t is a shortest path on Σ from y to x , and β_t is a shortest path on Σ from $f(x)$ to y .

Let $X_n(\Sigma)$ be the configuration space of all ordered n -tuples of pairwise distinct points in Σ . Its fundamental group $\pi_1(X_n(\Sigma), z)$ is identified with the surface pure braid group $P_n(\Sigma)$, where $z = (z_1, \dots, z_n)$ in $X_n(\Sigma)$ is a base point. For almost every $x = (x_1, \dots, x_n) \in X_n(\Sigma)$ the n -tuple of loops $(\gamma_{x_1, z_1}, \dots, \gamma_{x_n, z_n})$ is a based loop in the configuration space $X_n(\Sigma)$. Let $\gamma(\{f_t\}, x) \in P_n(\Sigma) = \pi_1(X_n(\Sigma), z)$ be the element represented by this loop, and let $\varphi: P_n(\Sigma) \rightarrow \mathbf{R}$ be a homogeneous quasimorphism. If Σ is a hyperbolic surface then the center $Z(P_n(\Sigma))$ is trivial, and hence the braid $\gamma(\{f_t\}, x)$ does not depend on the choice of the center $Z(P_n(\Sigma))$. If $\Sigma = T^2$, and φ vanishes on $Z(B_n(T^2)) \cong \mathbb{Z}^2$ then the number $\varphi(\gamma(\{f_t\}, x))$ does not depend on the choice of the isotopy $\{f_t\}$. In both cases from now on $\varphi(\gamma(\{f_t\}, x))$ is denoted by $\varphi(\gamma(f, x))$. Similarly to the spherical case, for $f, g \in \mathcal{G}_\Sigma$ we have the following cocycle condition:

$$(6) \quad \varphi(\gamma(gf, x)) = \varphi(\gamma(g, f(x))) \cdot \varphi(\gamma(f, x)).$$

Let L be an equator in Σ and D_L be an open disk in Σ whose boundary is L . Let the subspace $X_n(D_L \cup (\Sigma \setminus D_L)^\circ) \subset X_n(S^2)$ consisting of those tuples of n points in Σ , such that either all points in such a tuple lie in D_L , or all points in such a tuple lie in $(\Sigma \setminus D_L)^\circ$, or exactly $n-1$ points lie in D_L and one point in $(\Sigma \setminus D_L)^\circ$, or exactly $n-1$ points lie in $(\Sigma \setminus D_L)^\circ$ and one point in D_L . Let $\varphi \in Q_{\text{BF}}(B_n(S^2))$ a nontrivial homogeneous quasimorphism. We define $\Phi_{\Sigma, n}: \mathcal{G}_\Sigma \rightarrow \mathbb{R}$ as follows:

$$(7) \quad \Phi_{\Sigma, n}(f) := \int_{X_n(\Sigma^2)} \varphi(\gamma(f, x)) dx - \int_{X_n(D_L \cup (\Sigma \setminus D_L)^\circ)} \varphi(\gamma(f, x)) dx.$$

Now, the proof of Properties 1, 3 and 4 is identical to the proof of these properties in the spherical case. The proof of Property 2 is very similar to the spherical case as well. The key idea, similar to the spherical case, is to use the Ishida construction for \mathcal{G}_Σ presented in [5], see also [3, 4]. We leave the details for the interested reader and obtain a proof of Theorem B.

Remark 6. Theorem B is also true for an orientable surface of any genus with non-trivial boundary. Moreover it applies to “diameters” in the surface, defined as embedded paths with endpoints on the boundary, which are contractible relative to the boundary (in the case of the disk and Hofer’s metric the geometry of this space was studied in [16]). The proof goes along the same lines and is left to the reader.

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DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, ISRAEL

Email address: `brandens@bgu.ac.il`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MONTREAL, C.P. 6128 SUCC. CENTRE-VILLE MONTREAL, QC H3C 3J7, CANADA

Email address: `egor.shelukhin@umontreal.ca`